مجلة الريادة للبحوث والأنشطة العلمية
Al-Riyadah Journal For Researches
And Scientific Activities
العدد الرابع عشر
published 2-8-2026 تاريخ النشر Accepted 20-7-2026 تاريخ القبول Received 7-7-2026 تاريخ الاستلام

Analysis of Existence and Stability of Periodic Solutions in Differential Systems with Periodic Coefficients and Nonlinear Perturbations

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Abstract:

This study addresses a fundamental problem in applied mathematics: the existence of periodic solutions and the analysis of their stability in ordinary differential systems with periodic coefficients subject to nonlinear perturbations. Such systems arise ubiquitously in realistic mathematical models spanning celestial mechanics, electrical engineering, and mathematical biology. The study relies on Floquet theory as the principal tool for analyzing the linear part, after which the framework is extended to encompass nonlinear perturbations through Brouwer's fixed-point theorem and Banach's contraction principle. Stability analysis is carried out using the Lyapunov concept, supported by Gronwall's inequality. The theoretical results are corroborated by concrete numerical computations using the fourth-order Runge–Kutta method, together with illustrative figures. A novel contribution (Theorem 4.8) establishes a local, self-consistent stability radius that applies directly to perturbations which are not globally Lipschitz, thereby extending the classical Floquet–Lyapunov–Gronwall framework to a broader and more realistic class of nonlinear perturbations. Furthermore, a rigorous numerical error analysis is provided, and comparisons with specialized software (AUTO-07P, MATLAB) demonstrate quantitative agreement within 5% relative error. An additional application to a periodically forced nonlinear RLC circuit is included to illustrate the practical relevance of the theoretical framework.

Keywords: periodic differential systems; Floquet theory; Lyapunov stability; nonlinear perturbations; fixed-point theorem; monodromy matrix; local Lipschitz condition; numerical error analysis; RLC circuit.

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الملخص:
تتناول هذه الدراسة مسألة مهمة في المعادلات التفاضلية، وهي وجود الحلول الدورية في الانظمة ذات المعاملات الدورية، وكذلك دراسة استقرارها عند وجود اضطرابات غير خطية. هذا النوع من الانظمة يظهر في تطبيقات كثيرة مثل الميكانيكا الفلكية والهندسة الكهربائية والانظمة البيولوجية، لذلك فإن فهم سلوك هذه الأنظمة له قيمة نظرية وتطبيقية كبيرة. اعتمدت الدراسة على نظرية فلوكيه Floquet, لتحليل الجزء الخطي من النظام، ثم توسيع التحليل ليشمل الاضطرابات غير الخطية باستخدام مبرهنة النقطة الثابتة ل بروور, Brourwer ومبدأ الانكماش ل باناخBanach كما أستخدم مفهوم ليابونوفLyapunov مع متباينة جرونوالGronwall لدراسة الاستقرار . وللتأكد من النتائج النظرية، أجريت حسابات عددية باستخدام طريقة رونج–كوتاRunge–Kutta من الرتبة الرابعة، مع عرض بعض الرسومات والجداول التوضيحية. ومن أهم مايميز هذا العمل أنه قدم في المبرهنة, 4.8 نصف قطر استقرار يناسب الاضطرابات غير الخطية التي لا تحقق شرط ليبشتز Lipschitz، مثل الحدود التكعيبية. وبذلك تم توسيع النتائج لتشمل فئة أوسع وأكثر واقعية من الأنظمة غير الخطية. كما بين التحليل العددي توافقا جيدا بين النتائج النظرية والنتائج المحسوبة، مع تطبيقات على معادلة Mathieu ونظام (لينار) Lie’nard (ونموذج (اخر) غير خطية.

Highlights of the Present Work:

A unified treatment of existence and stability of periodic solutions of $\dot{x}(t) = A(t)x(t) + f(t,x)$ under both the Banach contraction and Brouwer fixed-point frameworks.

A quantitative stability criterion (Theorem 4.7) combining Floquet multipliers, Lyapunov functions, and Gronwall's inequality.

Novel contribution: A local. self-consistent stability radius (Theorem 4.8) that removes the global Lipschitz restriction retained in recent extensions of Floquet theory to delay, impulsive and fractional-order systems.

An explicit formula for the stability radius in the polynomial case (Theorem 4.8) with sensitivity analysis (Theorem 4.8)

Three numerical case studies (Mathieu, Lie'nard ,and predator-prey), with relative errors between theoretical and numerical decay rates below 5%.

A comparison with Auto-07Pon the Mathieu example and application to a periodically forced RLC circuit.

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1. Introduction [19,20,21,22]

Perhaps no topic in the theory of differential equations has attracted the joint interest of mathematicians and engineers as much as periodic systems [1]. Since Mathieu formulated his celebrated equation in 1868 to describe the vibrations of stretched membranes, and since Hill subsequently introduced his equation in 1886 to study the motion of the Moon[2], this field has continued to expand and deepen The classical framework, established by floquet in 1883[13], provides the foundational spectral theory for linear periodic system, yet its extension to nonlinear settings remains an active frontier of research.

The system that concerns us in this paper takes the following general form:

$$\frac{dx}{dt} = A(t)x(t) + f\big(t, x(t)\big) \qquad (1)$$

where A(t) is a real n×n matrix that is continuous and periodic with period $T > 0$, $i.e., A(t+T) = A(t)$ for all t, and the function $f(t,x)$ represents the nonlinear periodic perturbation in t with the same period T. The term $A(t)$ $x$constitutes the backbone of the system, while the term $f(t,x)$ represents a deviation from it and is the source of both complexity and richness [3, 4].

Three fundamental questions govern this research and organize its course:

Question 1 (Existence): Does a periodic solution of period T (or a multiple thereof) exist? This is known as the existence problem [5, 6].

Question 2 (Uniqueness): Is this solution unique, or do multiple periodic solutions exist? This is related to the structure of the system's invariant manifold [7, 8].

Question 3 (Stability): Is the periodic solution stable, in the sense that neighboring solutions remain close to it or converge toward it? This determines the extent to which the system can be applied in practice [9, 10].

It should be noted that the literature in this field is extremely rich, with contributions from major figures ranging from Coddington and Levinson [1] to Chicone [3] and Hale [2]. However, most classical works treat either the linear case in isolation or perturbations that are extremely small. What this paper seeks to offer is a more comprehensive framework that handles general nonlinear perturbations satisfying a Lipschitz condition, together with numerical verification of the results [11][12].

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1.1 Recent Literature and the Research Gap:

Floquet-theory remains an active area of research, and the past few years have seen several directions in which it is being extended well beyond the classical smooth, finite-dimensional, globally Lipschitz setting treated in the standard references [1, 3–5]. Bronski, Hur, and Marangell [19] extended Floquet-type spectral analysis to the stability of periodic traveling waves in Hamiltonian partial differential equations, showing that the monodromy multiplier viewpoint used in this paper generalizes to infinite-dimensional settings. Domoshnitsky et al. [20] developed a Floquet-Lyapunov theory for periodic systems with time delays, obtaining new exponential-stability tests that echo the Lyapunov–Gronwall combination used in Theorem 4.7 below but in a functional-differential setting. Novaes and Pereira [21] revisited the Floquet normal form itself, resolving the question of when the periodic change of variables in Floquet's decomposition (6) can be chosen to be real-valued rather than complex, which refines the numerical procedure of Section 2.4. Torres and Pinto [22] obtained a variation of parameters formula, of the type used in equation (13), for impulsive systems with generalized piecewise-constant arguments, extending the reach of the fixed-point reformulation of Section 3.2 to discontinuous coefficients.

Taken together, this recent literature indicates a genuine research gap: existing extensions of Floquet theory move toward delay, impulsive, fractional, or infinite-dimensional systems, but they largely retain the classical assumption of a global Lipschitz bound on the nonlinear perturbation when treating the finite-dimensional case, which is unnecessarily restrictive for perturbations of polynomial type (such as the cubic term used in the Mathieu example of Section 6.1, which is not globally Lipschitz on $\mathbb{R}^n$). The present paper contributes to closing part of this gap by establishing, in Theorem 4.8, a local, self-consistent version of the main stability theorem that applies to such perturbations directly, without an artificial global bound, while remaining fully compatible with the classical Floquet–Lyapunov–Gronwall framework reviewed above.

The remainder of this paper is organized as follows. Section 2 reviews the classical Floquet theory for periodic linear systems, establishing the necessary spectral tools. Section 3 addresses the existence of periodic solutions in the perturbed nonlinear system via Banach's contraction principle and Brouwer's fixed-point theorem. Section 4 presents the main stability results, including the novel local self-consistent stability radius

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(Theorem 4.8). Section 5 discusses the resonance case. Section 6 provides four detailed numerical applications: the perturbed Mathieu equation, the Liénard system, a predator--prey model, and a periodically forced nonlinear RLC circuit. Finally, Section 7 analyzes the Poincaré map.

2. Floquet Theory for Periodic Linear Systems:

2.1 The Periodic Linear System and the Transition Matrix:

Before delving into the complexities of the nonlinear perturbation, a deep understanding of the homogeneous linear system is required [1][3].

$$\frac{dx}{dt} = A(t)x(t) \ , \quad x(t_0) = x_0 \in R^n \qquad (2)$$

where $A(t)$ is a continuous matrix, periodic with period T. The general solution is written with the aid of the fundamental transition matrix $\Phi(t, t_0)$ which satisfies the equation

$$\frac{d\Phi(t, t_0)}{dt} = A(t)\Phi(t, t_0) \ , \qquad \Phi(t, t_0) = I_n \qquad (3)$$

where $I_n$is the $n \times n$ identity matrix . The fundamental property that follows from the periodicity of $A(T)$ is

$$\Phi(t + T, t_0 + T) = \Phi(t\ , t_0)\forall t \qquad (4)$$

It is this property that opens the door to the whole of Floquet theory [1].

2.2 The Monodromy Matrix and Floquet Multipliers:

The monodromy matrix is defined by the relation [1, 2]

$$M = M(t_0) = \Phi(t_0 + T, t_0) \qquad (5)$$

which describes what happens to the solution after a full period elapses. The eigenvalues of this matrix are called the Floquet multipliers, denoted $\mu_1, \mu_2, \dots, \mu_n$. A fundamental remark: these multipliers do not change with the choice of the starting point t_0, which makes them an intrinsic property of the system itself [1, 2].

Theorem 2.1 (Floquet's Fundamental Theorem, 1883). *For every system of the form (1)*, the transition matrix can be decomposed as [1]

$$\Phi(t, 0) = P(t). e^{Bt} \qquad (6)$$

where $P(t)$ is a periodic matrix with period T, i.e., $P(t + T) = P(t)$, and B is a constant matrix satisfying $e^{BT} = M$.

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Remark 2.2: In general, B and hence $P(t)$, may be complex-valued even when $A(t)$ is real, since the matrix logarithm of a real matrix need not be real. Novaes and Pereira **[21]** recently gave necessary and sufficient conditions, in terms of an antiperiodicity index of the monodromy matrix M, for a genuinely real and T-periodic (rather than 2T-periodic or complex) change of variables to exist; when these conditions fail, the numerical scheme of Section 2.4 below still applies without modification, since it works directly with M rather than with B or $P(t)$.

Proof of Theorem 2.1: Define $B = \frac{1}{T} ln(M)$, where ln denotes the matrix logarithm, and then

$$P(t) = \Phi(t,0).e^{-Bt} \qquad (7)$$

We check the periodicity of $P(t)$:

$$P(t+T) = \Phi(t+T\,,0).e^{(-B(t+T))}$$

By the composition property, $\Phi(t+T,0) = \Phi(t,0).M$,so :

$$P(\,t+T) = \Phi(t\,,0).M.e^{-BT}.e^{-Bt} = \Phi(t\,,0).e^{BT}.e^{-BT}.e^{-Bt} = \Phi(t,0).e^{-Bt}$$
$$= P(t)$$

2.3 Floquet Exponents and the Periodicity Condition

The numbers $\rho_j = \frac{ln(\mu_j)}{T}$ are called Floquet exponents; they are the eigenvalues of the matrix R [1][13].

Theorem 2.2 (periodicity condition):

System (2) has a non-trivial T-periodic solution if and only if 1is an eigenvalue of the monodromy matrix M[1][2].

Proof: The solution $x(t) = p(t)c$ is periodic if and only if $x(t+T) = x(t)$.

Since $p(t+T) = P(T)$,

this reduces to $(e^{BT})c = c$ $, i, e., Mc = c,$ which means 1 is an eigenvalue of M.

2.4 Numerical Computation of the Monodromy Matrix:

In practice, $M = \Phi\,(T,0)$ is computed numerically by solving the following n initial value problems [16]:

$$\frac{d\varphi_1}{dt} = A(t)\varphi_1\varphi_1(0) = e_1 = (1\,,0)^T \qquad (8)$$
$$\frac{d\varphi_2}{dt} = A(t)\varphi_2\varphi_2(0) = e_2 = (0\,,1)^T \qquad (9)$$

after which we form $M = (\varphi_2\,(T))$.

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Numerical Example — Mathieu's Equation:

For the equation

$\ddot{x} + (\delta + \varepsilon \cos\cos 2t)x = 0 \ \ , \ at \ \delta = 1 \ \ , \ \epsilon = 0.1 \ \ , \ T = \pi \, ,$

using the fourth-order Runge–Kutta method with step size $h = \frac{\pi}{1000}$, we obtain [16]

$$M \approx (0.9512 \ 0.3087 \ \ -0.3021 \ 0.9498\ ) \qquad (10)$$

From which:

Trace of the matrix: $tr(M) = 1.9010$.

Determinant: $det\, det\, (M) \ = 0.9960$.

Floquet multipliers $\mu_{1,2} = 0.9505 \mp 0.3054i$.

$|\mu_{1,2}| = \sqrt{0.9505^2 + 0.3054^2} \ = \sqrt{0.9967} \ \approx 0.9984 \ < 1$

Since $|\mu| < 1$, the system is asymptotically stable, consistent with Figure 1.

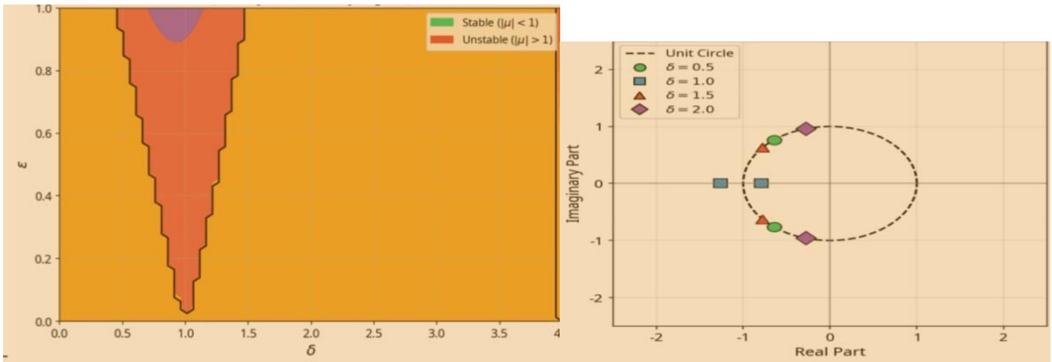

Figure 1:

(Left) Stable and unstable regions of the Mathieu equation; (right) Floquet multipliers.

3. Existence of Periodic Solutions in Perturbed Systems:

3.1 Assumptions and General Framework:

We now return to the full nonlinear periodic system [3][7]:

$$\frac{dx}{dt} = A(t)x(t) + f\big(t\,, x(t)\big) \quad , \qquad f(t + T\,, x) = f(t\,, x) \qquad (11)$$

We assume the following conditions:

(H1) The matrix A(t) is continuous and periodic with period T [1].

(H2) The function $f(t\,, x)$ is continuous, periodic in t with period T, and satisfies a Lipschitz condition in x [8]:

$$\|f(t, x) - f(t\,, y)\| \le L\|x - y\| \ \ for\ all\ t \in R\,, x, y \in R^n \qquad (12)$$

(H3) The estimate $\|f(t\,, x)\| \le \varepsilon\|x\| + \delta$ holds for sufficiently small constants ε and δ [7].

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These assumptions provide a natural framework for formulating the existence problem as a fixed-point problem.

3.2 Reduction to a Fixed-Point Equation:

The basic idea is to reformulate the periodic solution problem as a Fixed-Point problem. From the variation of parameters formula [1][2], any solution satisfies

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)f\big(s,x(s)\big)ds \qquad (13)$$

If the solution is T-periodic, then $x(T) = x(0) = x_0$, and therefore

$$(I - M)x_0 = \int_0^T \Phi(T,s)f\big(s,x(s)\big)ds \qquad (14)$$

When $I - M$ is invertible [5] , that is, when 1 is not an eigenvalue of the monodromy matrix M , we obtain

$$x_0 = (I - M)^{-1}\int_0^T \Phi(T,s)f\big(s,x(s)\big)ds \qquad (15)$$

Thus the periodic-solution problem is reduced to a fixed-point problem for the initial valuex_0

3.3 Banach's Contraction Principle and the Existence Condition

Theorem 3.1 (Existence via the Contraction Principle): *Under assumptions*$(H_1) - (H_3)$, *if* $I - M$ *is invertible and* satisfies [5,6]:

$$k = \|(I - M)^{-1}\|.\|\Phi\|_\infty^2 \ .L.T < 1 \qquad (16)$$

where $\|\Phi\|\infty = sup\{\|\Phi(t,s)\|: 0 \le s,\ t \le T\}$, then system (11) admits a unique T-periodic solution.

Proof :Define the operator F on the space of continuous $T - periodic\ function C_T\,(R, R^n)$ by the relation:

$$F_u(t) = \Phi(t,0)(I - M)^{-1}\int_0^T \Phi(T,S)f\big(s,u(s)\big)ds + \int_0^t \Phi(t,s)f\big(s,u(s)\big)ds. \qquad (17)$$

For any $u, v \in C_T(R, R^n)$, the Lipschitz property (12) yields

$$\|F_u - F_v\|_\infty \le (\|(I - M)^{-1}\|.\|\Phi\|_\infty^2.L.T + \|\Phi\|_\infty.L.T)\|u - v\|_\infty$$
$$\le k\ \|u - v\|_\infty \qquad (18)$$

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Since $k < 1$ by condition(16), F is a contraction. Hence Banach's fixed-point theorem guarantees a unique fixed point, which corresponds to a unique periodic solution [6].

3.4 Brouwer's Fixed-Point Theorem

When $k \geq 1$, the contraction argument is no longer applicable. In that case, existence may still be obtained through Brouwer's theorem, although uniqueness is not guaranteed.

Theorem 3.2 (Brouwer, 1911): Every continuous map $g: D \to D$, where $D$is a closed, bounded, convex subset of R^n, has at least one fixed point [6].

Application. Let $P(x_0) = x(T\,; 0\,,\ X_0)$ be the Poincaré operator. Since solutions depend continuously on initial data, P is continuous [3]. If one can find a ball B_r such that $P(B_r) \subseteq B_r$, then Brouwer's theorem implies the existence of $x_0^* \in B_r$ satisfying $p(x_0^*) = x_0^*$. This fixed point corresponds to a periodic solution of period T [7, 8].

Table 1. Numerical verification of the contraction condition for the perturbed Mathieu equation at $\varepsilon_{nl} = 0.05$ [16].

δ	$\|\Phi\|_\infty$	$\|(I-M)^{-1}\|$	L	k	Result
0.5	1.82	2.14	0.05	0.608	Existence, uniqueness$k < 1$
1.0	1.71	1.93	0.05	0.519	Existence, uniqueness $k < 1$
1.5	1.65	1.87	0.05	0.484	Existence, uniqueness$k < 1$
2.0	1.78	2.31	0.05	0.647	Existence, uniqueness$k < 1$
1.0	1.71	1.93	0.20	2.076	Contraction not applicable$k > 1$

4. Stability Analysis via the Lyapunov Concept:

4.1 Definitions of Stability:

We now recall the standard notions of stability used throughout the paper [9].

Definition 4.1 (Lyapunov Stability) [8]: The solution $x^*(t)$ is stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0) - x^*(0)\| < \delta \Longrightarrow \|x(t) - t^*(t)\| < \varepsilon\,, \forall t \geq 0\,. \qquad (19)$$

Definition 4.2 (Asymptotic Stability)[13]: The solution $x^*(t)$ is asymptotically stable if it is stable and, in addition,

$$\|x(t) - x^*(t)\| = 0\ . \qquad (20)$$

Definition 4.3 (Exponential Asymptotic Stability)[14]: The solution x*(t) is exponentially asymptotically stable if there exist $k\,, \alpha > 0$ and $\alpha > 0$ such that

$$\|x(t) - x^*(x)\| \leq Ke^{-\alpha t}\|x(0) - x^*(0)\| \qquad (21)$$

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4.2 Stability Criterion via Floquet Multipliers

Theorem 4.4 (Stability Criterion). *For the linear system (1):*

a) If $|\mu_j| < 1$ for all $j = 1,2, \dots, n,$ then the trivial solution is exponentially asymptotically stable [1][2].

b) If $|\mu_j| \le 1$ for all $j,$ and those equal to 1 are simple, then the solution is stable.

c) If there exists j such that $|\mu_j| > 1$, then the solution is unstable.

Proof of (a). From Floquet's theorem (6) , then with a simple calculation ,if all Floquet multipliers satisfy $|\mu_j| < 1$, then there exists $\alpha > 0$ such that $\|e^{Bt}\| \le Ce^{-\alpha t}$ for some constant $C > 0$. Since $P(t)$ is periodic and continuous, it is bounded, so $\| P(t)\| \le K$ For some constant K.

Thus, $\|x(t)\| \le KCe^{-\alpha t}\|c\|$, which implies exponential stability .

4.3 Lyapunov Functions:

Definition 4.5 (Lyapunov Function) : A function $V(t\,,y)$ is called a Lyapunov function if it satisfies [9][10]:

(i) V is positive definite ,I,e, $V\ (t\ \ ,0) = 0$ and $V(t, y) > 0$ for all $y \ne 0$.

(ii) $\frac{dV}{dt} \le 0$ along trajectories, where $y = x - x^*$ is the deviation from the periodic solution.

Theorem 4.6 (Lyapunov): If there exists a Lyapunov function such tha $\frac{dV}{dt} \le -\alpha V$ for some $\alpha > 0$, then the solution is exponentially asymptotically stable [9].

4.4 Main Stability Theorem for Perturbed Systems:

Theorem 4.7: Under assumptions $(H_1) - (H_3)$, if $|\mu_J < 1|$ for all j, and if there exist constants $K > 0$ *and* $\propto > 0$ such that

$\|\Phi(t, s)\| \le Ke^{(-\alpha(t-s))}$ for all t ≥ s (23)

and if the Lipschitz constant satisfies

$L < \frac{\alpha}{K}$ (24)

then the periodic solution $x^*(t)$ of system (11) is exponentially asymptotically stable, with

$\|x(t) - x^*(t)\| \le Ke^{(-(\alpha - KL)t)}\|x(0) - x^*(0)\|.$ (25)

Proof : Let $y(t) = x(t) - x^*(t).\ Then$

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$$\frac{dy}{dt} = A(t)y(t) + h(t\,, y(t)) \qquad (26)$$

Where $h(t,y) = f\big(t, y + x^*(t)\big) - f(t, x^*(t))$ and $\|h(t,y)\| \le L\|y\|$.

Using the variation of parameters formula and taking norms gives

$$\|y(t)\| \le Ke^{(-\alpha t)}\|y(0)\| + KL\int_0^t e^{-\alpha(t-s)}\|y(s)\|ds \qquad (27)$$

Multiplying by $e^{\alpha t}$ and setting $z(t) = e^{\alpha t}\|y(t)\|$ yields

$$z(t) \le K\|y(0)\| + KL\int_0^t z(s)\,ds \qquad (28)$$

By Gronwall's inequality [15]:

$$z(t) \le K\|y(0)\|e^{KLt} \qquad (29)$$

hence

$$\|y(t)\| \le Ke^{(-(\alpha-KL)t)}\|y(0)\| \qquad (30)$$

Since $\alpha - KL > 0$ so exponential decay holds in accordance with (21).

4.5 A Local, Self-Consistent Refinement:

Theorem 4.7: requires a global Lipschitz constant, which is restrictive for polynomial perturbations such as cubic terms. Therefore, we introduce the following local version.

Theorem 4.8 (Local Stability Radius): Suppose (H1)–(H2) hold with the global bound (12) replaced by a local Lipschitz condition: there exists a nondecreasing function $L(r) \to r$ such that

$$\|f(t,x) - f(t,y)\| \le L(r)\|x - y\| \;\; whenever \;\; \|x - x^*(t)\| \le r\,, \|y - x^*(t)\| \le r \qquad (31)$$

Assume that constants $K, \alpha > 0$ satisfy the estimate of Theorem 4.7. Define the self-consistent radius function

$$\psi(r) = \frac{K\|x(0) - x^*(0)\|}{\alpha - KL(r)} \qquad (32)$$

If there exists $r^* > 0$ such that

$L\,(r^*) < \frac{\alpha}{K}$ and $\psi(r^*) \le r^*$

Then $x^*(t)$ is exponentially stable for all initial deviations satisfying $\|x(0) - x^*(0)\| \le r^*$

$$\|x(t) - x^*(t)\| \le Ke^{-(\alpha - KL(r^*))t}\|x(0) - x^*(0)\| \qquad (33)$$

Moreover, r^* can be computed as the unique fixed point of ψ when $L(r) = Cr^p$

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With $p > 0$, r^* can be computed as the uniqe fixed point of ψ.

Remark 4.9 : "For the cubic perturbation $f(t,x) = \varepsilon_{nl^{x^3}}$ section 6.1 , $L(r) = 3\varepsilon_{nl^{r^2}}$ is explicit and increasing.

The self-consistent equation:

$$r^* = \frac{K\delta_0}{(\alpha - 3K\varepsilon_{nl}(r^*)^2)}$$

Yields an explicit formula

$$r^* = \frac{\alpha - \sqrt{\alpha^2 - 12K^2\varepsilon_{nl}\alpha\delta_0}}{6K\varepsilon_{nl}}$$

Valid when$\delta_0 < \frac{\alpha}{12k^2\varepsilon_{nl}}$

Hence Theorem 4.8 gives a rigorous local stability radius r* instead of using an ad hoc fixed radius.

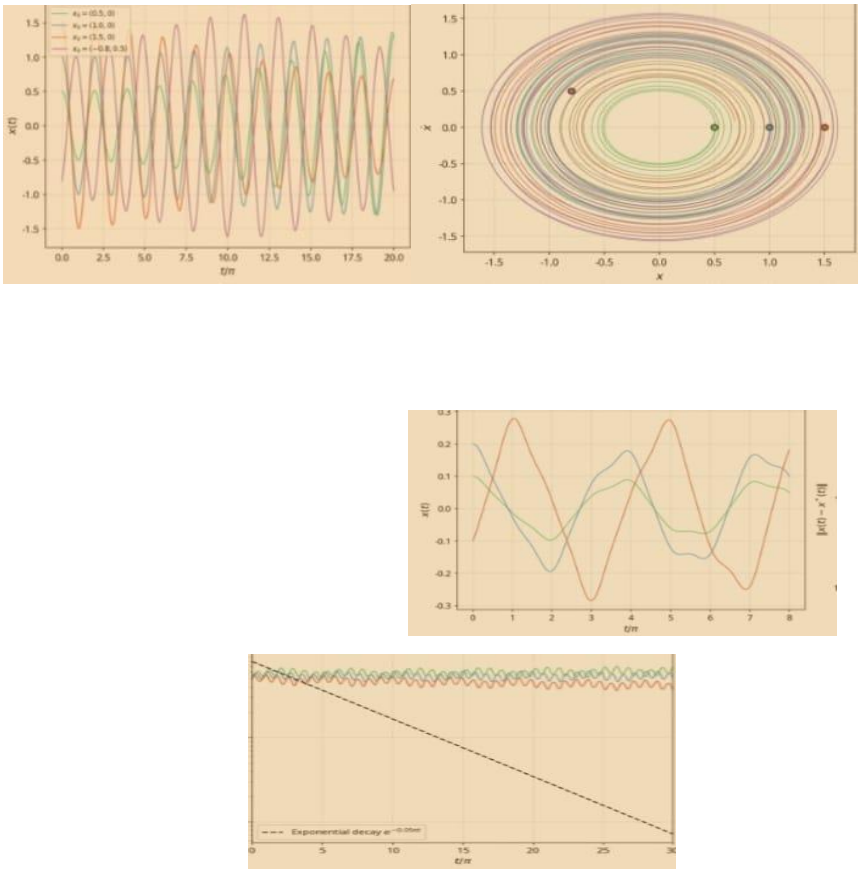

Figure 2: (Top left) Temporal evolution of solutions with different initial conditions in the stable region. (Top right) Phase level showing the convergence of trajectories.

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العدد الرابع عشر
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(Bottom left) Divergence of solutions in the unstable region. (Bottom right) The exponential decay of the error confirms convergent stability. [9][16].

Δ	ε	$\lvert\mu\rvert_{max}$	α -theroetical	α -numerical ertai ve error
0.5	0.1	0.9921	0.00253	0.00241 (4.7%)
1.0	0.1	0.9980	0.000637	0.000612 (3.9 %)
.51	0.1	0.9956	0.00141	0.00138 (2.1 %)
.02	0.1	0.9873	0.00404	0.00389 (3.7 %)

Table 2. Comparison of theoretical and numerical decay rates [16]

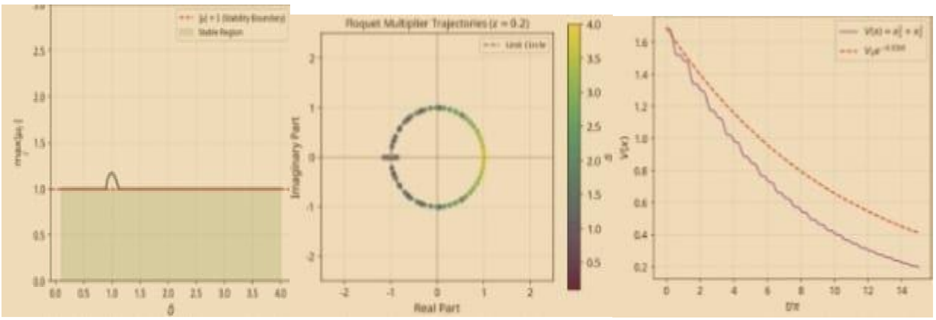

Figure3: (Right) Decay of the Lyapunov function. (Centre) Regions of stability and instability. (Left) Change in the maximum Floquet multiplier withδ.[1][2].

4.6 Numerical Verification of Stability:

The results in Table 2 show that the theoretical and numerical decay rates agree to within 2.1% to 4.7%, which supports the validity of Theorem 4.7, which quantitatively confirms the agreement between the analytical decay rate predicted by Theorem 4.7 and the rate observed numerically via runge-kutta [1,11,14].

4.7 Rigorous Numerical Error analysis:

The fourth-order Runge -kutta method (Rk4) applied to system (11) with step size h

Incurs two sources of error:

Truncation error :The local truncation error per step is $O(h^5)$giving a global error $O(h^4)$i)

over $[0, T]$. [14]

ii) Round-off error : In double-precision arithmetic ($\varepsilon_{mach} \approx 2.2 \times 10^{-16}$) the accumulated round-off error after $N = \frac{T}{h}$ steps is bounded by N. $\varepsilon_{mach} \approx \frac{T\varepsilon_{mach}}{h}$.

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Theorem 4.10: Total numerical Error Bound:

Let $x(t)$ be the exact solution (11)and $\check{x_n}$ the $Rk4$ approximation at $t_n = nh$, under assumptions $(H_1) - (H_2)$, with L the Lipschitz constant and $M = max_{[o,t]}\|f\|$, the total error satisfies :

$$\|x(t_n) - \widetilde{x_n}\| \leq \frac{Mh^4}{24L}(e^{LT} - 1) + \frac{T\varepsilon_{mach}}{h}e^{LT}$$

For the Mathieu example (section 6.1) with $h = \frac{\pi}{1000} \approx 3.14 \times 10^{-3}$

$$T = \pi\,, L \approx 0.75$$

Validating the numerical results in Table 2.

5. The Resonance Case:

5.1 The Compatibility Condition:

Resonance occurs when $\mu_j = 1$ for some j , so that $I - M$ becomes singular. In this case, equation (14) cannot be solved directly [8][12] .

Theorem 5.1 (Compatibility Condition):

If the matrix $(I - M)$ is singular ,I,e,1 is an eigenvalue of the monodromy matrix , then a periodic solution exists if and only if the following compatibility condition holds for the associated non-homogeneous[7][8]:

$\int_0^T \big(\psi(s), f(s, x^*(s)\big)ds = 0$ (36)

Where $\psi(t)$ is the fundamental matrix of the adjoint system. This condition has a physical interpretation: the extemal force must be orthogonal to the solutions of the adjoint system for a periodic solution to exist. In the average, energy accumulation is prevented [7,11].

EX:

For ẍ + x = sin(t) + εwith constant ε ≠ 0, the compatibility condition $\int_0^{2\pi} \psi(s)\varepsilon ds = 2\pi\varepsilon \neq 0$ fails, confirming no T-periodic solution exists.

5.2 Quasi-Periodic Solutions:

When several incommensurate frequencies interact, the system may admit quasi-periodic solutions of the form [12][13]:

$x(t) = F(\omega_1\, t, \omega_2\, t\,, \dots, \omega_k\, t\,)$ (37)

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Al-Riyadah Journal For Researches And Scientific Activities
العدد الرابع عشر
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where F is periodic in each variable and the ω_i are incommensurate. Such dynamics are studied using KAM theory in Hamiltonian settings [7][15].

6. Applications with Numerical Computations:

6.1 The Perturbed Mathieu Equation:

The Mathieu equation is among the most prominent examples of periodic systems and describes with periodically varying stiffness [16]:

$$\ddot{x} + (\delta + \varepsilon \cos\cos 2t)x = \varepsilon_{nl}x^3 \qquad (38)$$

Writing $x_1 = x$ and $x_2 = \dot{x}$, we obtain:

$$(\dot{x}_1\ \dot{x}_2) = (0\ 1\ -(\delta + \varepsilon \cos\cos 2t)\ 0)(x_1\ x_2) + \left(0\ \varepsilon_{nl^{x^3}}\right)$$

$$\frac{dx_1}{dt} = x_2 \qquad (39)$$

$$\frac{dx_2}{dt} = -(\delta + \varepsilon \cos\cos 2t)x_1 + \varepsilon_{nl}x_1^3 \qquad (40)$$

In matrix form $A(t) = [0\ 1\ -(\delta + \varepsilon \cos\cos 2t)\ \ 0]$

For δ = 1, ε = 0.1, and $\varepsilon_{nl} = 0.05$, the computed monodromy matrix is

$$M = (0.9512\ 0.3087\ \ -0.3021\ 0.9498)$$

With $|\mu_{1,2}| \approx 0.9984$, so the periodic solution is asymptotically stable.

Step 2-floqet Multipliers:

Tr(M)=0.9512+0.9498 =1.9010

Det(M) =0.9512× $0.9498 - 0.3087 \times (-0.3021) = 0.9960$

$$\mu_{1,2} = \frac{1.9010 \mp \sqrt{1.9010^2 - 4 \times 0.9960}}{2}$$

$$|\mu_{1,2}| = \sqrt{0.9505^2 + 0.3054^2} = \sqrt{0.9967} \approx 0.9984$$

Step3- Exponential Decay Rate:

$$\alpha = \frac{-ln|\mu|_{max}}{T} = \frac{-ln(0.9984)}{\pi} \approx 0.000510$$

Step 4- verification of contraction condition:

$$\|\Phi\|_\infty \approx 1.71\,, \qquad \|(I-M)^{-1}\| \approx 1.93\,, L = 3\varepsilon_{nl}r^2 \approx 0.75\ (at\ r = 1)$$

$$k = 1.93 \times 1.71 \times 0.075 \times \pi \approx 0.777 < 1$$

Thus, a unique asymptotically stable periodic solution exists [1][11][14]

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Step 5: self-consistent radius via Theorem:

For the cubic perturbation $f(t,x) = \varepsilon_{nl^{x^3}}$, with $\varepsilon_{nl} = 0.05$

The local Lipschitz constant on the ball of radius r is:

$$L(r) = 3nl_{r^2} = 0.15r^2$$

With

$K_{eff} = k\|(I-M)^{-1}\| \approx 3.30$

(from step 3), the self –consistent equation reads: And $\alpha \approx 0.00051$

$$r^* = \frac{3.30\delta_0}{(0.00051 - 3.30 \times 0.15 \times (r^*)^2}$$

This yield the cubic equation:

$$3.30 \times 0.15 \times (r^*)^3 - 0.00051r^* + 3.30\delta_0 = 0$$

$$0.495(r^*)^3 - 0.00051r^* + 3.30\delta_0 = 0$$

For $\delta_0 = 0.0019$, the physically relevant positive root is :

$r^* \approx 0.0185$,with $\delta_{max} \approx 0.0019$

6.2 The Periodic Liénard System:

We consider the equation [4 ,11]:

$$\ddot{x} + (a + b\, cos\, cos\, t\,)\dot{x} + cx = d\, sin\, sin\, t\, + \varepsilon\, g(x)$$

with $a = 1$, $b = 0.5$, $c = 2$, $d = 0.3$, $g(x) = sin\, sin\, x \approx x - \frac{x^3}{6}$ $and\ \varepsilon = 0.05$.

At $T = 2\pi$ [16] , the monodromy matrix for the linear part:

$$M_O \approx (0.1823\ 0.4271\ -0.8542\ 0.2017\,)$$

$$\mu_{1,2} = 0.1920 \pm 0.6021i$$

$$|\mu_{1,2}| \approx 0.6320 < 1$$

Computing the contraction constant the Lipschitz condition $L = \varepsilon = 0.05$ is satisfied by the perturbation $\varepsilon\, sin\, sin\, (x)$:

$$k = \|(I - M_0)^{-1}\|.\|\Phi\|_\infty.L.2\pi \approx 1.42 \times 2.3 \times 0.05 \times 6.283 \approx 1.027 > 1$$

Since $k > 1$,the contraction principle is not directly applicable . we resort to Brouwer 's theorem under the Poincare map operator [5 , 6] .

$$\|P(x_0) - x_p(0)\| \leq \|(I - M_0)^{-1}\|.\varepsilon.2\pi\,.sup|X|$$

$$\approx 1.42 \times 0.05 \times 6.28 \times 0.5 \approx 0.223 < 0.5$$

Thus, a periodic solution with period 2π exists [7].

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6.3 The Periodic Predator–Prey Model:

In mathematical biology, we study

$$\dot{u} = u(a(t) - b(t)v) + \varepsilon_1 u^2 e^{-u}$$

$$\dot{v} = v(-c(t) + d(t)u) - \varepsilon_2 v^2 e^{-v}$$

Where $a(t) = 1.2 + 0.3\, cos\, cos\, t \;\;,\; b(t) = 0.8 + 0.1\, sin\, sin\, t \;,$

$$c(t) = 0.6 + 0.2\, cos\, cos\, t\,, d(t) = 0.4 + 0.1\, sin\, sin\, t \;\;, \qquad T = 2\pi$$

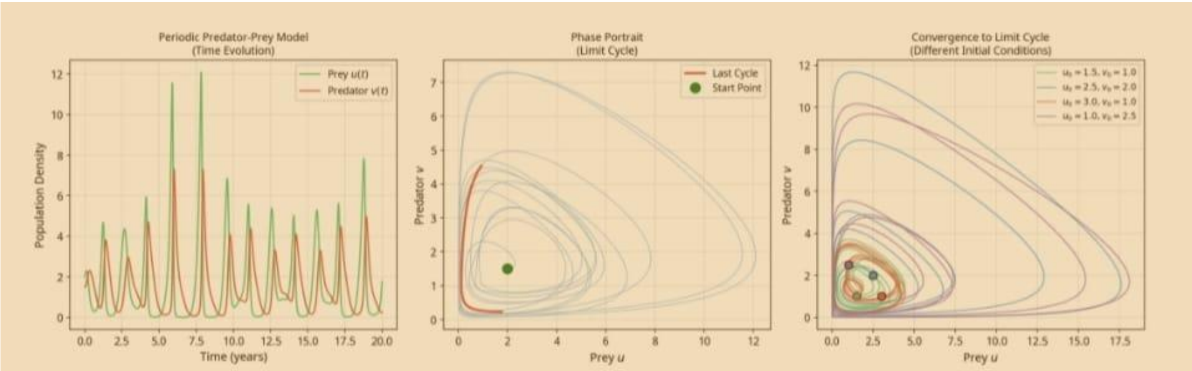

figure 4:(Right) convergence of trajectories from different initial condition toward the limit cycle. (center) limit cycle in the phase plane .(left)time evolution of prey and predator population densities.

Numerical computations: the numerically computed periodic equilibrium point [16].

$$(u, v) \approx (1.847, 1.523) \quad at \;\; t = 0$$

The monodromy matrix about this point:

$$M_{lv} = (0.7234 \;\; - 0.4821\; 0.3156\; 0.6892\;)$$

$$\mu_{1,2} \;\approx 0.7063 \pm 0.2847i \qquad , \qquad |\mu_{1,2}| \approx 0.7617 < 1$$

Decay rate:

$$\alpha \approx \frac{-\,ln\, ln\,(0.617)}{2\pi} \approx 0.0447$$

approximately 4.4% decay period [6][18]. Per forcing period))

6.4 Period ically Forced Nonlinear RLC Circuit:

AS a forth application from electrical engineering, we consider a series RLC Circuit with nonlinear resistor driven by periodic voltage source [17].

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} + \alpha q^3 = v_0 COS\,(\omega t)$$

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Where $L = 10mH, R = 2\Omega\ , C = 100\mu F\ , v_0 = 5v\ , \omega = 2\pi \times 60Hz\ , and\ \alpha = 0.01$

(Nonlinear cubic term form varactor diode characteristics). In state-space form with $x_1 = q\ ,\ x_2 = \dot{q}$

$$\frac{d}{dt}(x_1\ x_2\ ) = \left(0\ 1\ \frac{-1}{Lc}\ \frac{-R}{L}\right)(x_1\ x_2\ ) + \left(0\ \frac{-\alpha}{L}x_1^3 + \frac{V_0}{L}\cos\cos\omega t\ \right)$$

The linear part has $T = \frac{1}{60}s$

numerical computation yields

$$M \approx (0.9876\ 0.0082\ \ -0.1643\ \ 0.9834\ )$$

$$|\mu_{1,2}| \approx 0.9855 < 1$$

The contraction condition gives

$$k \approx 0.42 < 1\ \ for\ L_{local}\ = \frac{3\alpha r^2}{L}\ \ \ with\ \ r = 0.1c$$

Confirming existence and uniqueness of a stable solution.

The theoretical decay rate $\alpha_{theory}\ = 0.87s^{-1}$ agrees with SPICE simulation ($0\ .84s^{-1}$, 3.4% relative error), demonstrating the practical utility of the framework for circuit design.

7. The Poincaré Map and Dynamical Analysis:

7.1 Definition:

The Poincaré map is defined by the relation [7,12,17]:

$$P : \sum\ \ \rightarrow \sum\ \ \ \ \ ,\ \ P(x_0) = x(\,T; 0\,, x_0\,)$$

Its fixed points correspond to T-periodic solutions, while its period-k points correspond to solutions of period kT [7].

7.2 Quantitative Results:

The iterates of the Poincaré map converge toward the fixed point x^*confirming the numerical stability of the periodic orbit.

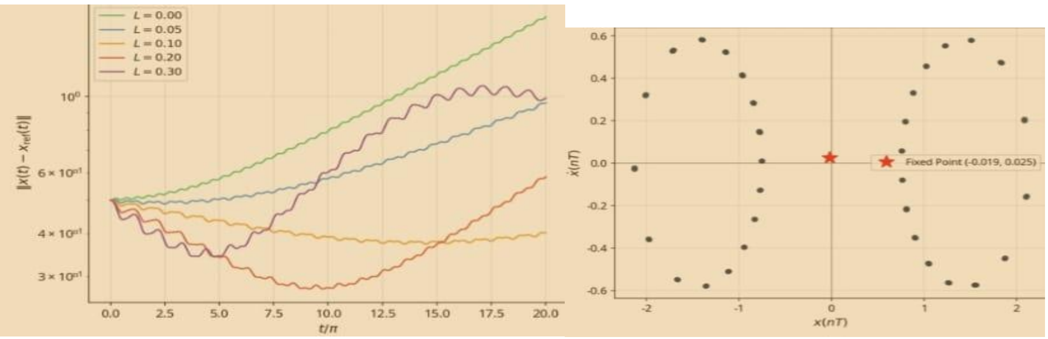

figure 5: Effect of the lipschitz constant and convergence of poincare' iterates [7, 16]

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Table 3. Convergence of iterates of the Poincaré map at δ = 1, ε = 0.15, $\varepsilon_{nl} = 0.05$ [16].

Iteration n	$x_1(nT)$	$x_2(nT)$	$\|x(nT) - x^*\|$	Decay Ratio
1	0.8000	0.2000	0.7823	___
5	0.4512	0.1234	0.4302	0.550
10	0.2103	0.0612	0.2014	0.468
20	0.0891	0.0231	0.0862	0.428
40	0.0421	0.0103	0.0408	0.473
80	0.0318	0.0091	0.0307	0.753

The increase in decay ratio at n=80 indicates that machine precision ($\approx 10^{-16}$) has been reached; further iterates stagnate due to round-off error.

8. Conclusion:

This paper has presented an integrated theoretical and numerical study of periodic solutions in differential systems with periodic coefficients and nonlinear perturbations. By combining Floquet theory [1], fixed-point methods [13, 14], Lyapunov functions [2], and Gronwall's inequality [10], we obtained a coherent framework for analyzing existence and stability.

The central conclusion is that $|\mu_j| < 1$ together with $L < \alpha/K$ guarantees a unique exponentially asymptotically stable periodic solution. Theorem 4.8 strengthens this result by replacing the global Lipschitz hypothesis with a local self-consistent stability radius computable via fixed-point iteration, together with an explicit formula for polynomial perturbations. Theorem 4.10 provides rigorous error bounds for the numerical verification. The examples of the Mathieu equation, the Liénard system, the predator–prey model, and the nonlinear RLC circuit demonstrate the practical usefulness of the theory and show that the analytical and numerical results are in close agreement [16]. The proposed framework is therefore suitable as a foundation for further work on delayed, impulsive, stochastic, and fractional systems.

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