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Monomial Bounds and Redundancy in Equivariant Schubert Calculus on Lagrangian Grassmannians

[1].AMNNAH ABUOJAYLAH ALHADI .. [2].NAJIMUDIN AMIR MAWLOUD ..[3]. WEDYAN MUKTAR A ELJIAFA (1+3) Faculty of Education / University of Zawiya. Libya..(2) Faculty of Science / University of Zawiya, Libya

[1]am.rasheed@zu.edu.ly [2]N.Shalghoum@zu.edu.ly [3] w.eljiafari@zu.edu.ly

https://orcid.org/0009-0001-3676-4673 .. https://orcid.org/0009-0004-7747-9969 .. https://orcid.org/0009-000-8747-5970

Abstract:

We prove an unconditional upper bound for the number of distinct monomials in any equivariant Schubert coefficient on $LG(n, 2n)$:

$$|Mon(C^{\nu}_{\lambda,\mu})| \le \binom{n+d-1}{d},$$

where $d = |\lambda| + |\mu| - |\nu|$. The bound is sharp and follows solely from the polynomial nature of the coefficient. We introduce a conjectural edge-labeled tableau model (Conjecture 3.1) for the Lagrangian Grassmannian. Under this conjecture, we obtain a complementary bound $|Mon| \le |\mathcal{T}(\lambda, \mu, \nu)|$ and the refined estimate

$|Mon(C^{\nu}_{\lambda,\mu})| \le min\left(|\mathcal{T}(\lambda, \mu, \nu)|, \binom{n+d-1}{d}\right)$.

We define a redundancy index $R(m) = |\Phi^{-1}(m)|$ and prove, via the pigeonhole principle,

$\max_m R(m) \ge \left\lceil \frac{|\mathcal{T}|}{|Mon|} \right\rceil$.

When $|\mathcal{T}| > \binom{n+d-1}{d}$, the bound forces the maximum redundancy to be at least 2. We characterize injectivity of the weight map and show that the conjectural model recovers the classical Tamvakis rule [5]. A detailed example for $n = 3$ illustrates the theory.

Keywords: Equivariant Schubert calculus; Lagrangian Grassmannian; edge-labeled tableaux; monomial bounds; strict partitions; redundancy index.

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الملخص:

حدود أحادية وتكرار في حساب شوبرت التكافئي على غراسمانيان لاغرانج

نبرهن حداً أعلى غير شرطي لعدد الحدود الأحادية في أي معامل شوبرت تكافئي على $LG(n,2n)$

$$\left|Mon\left(C_{\lambda,\mu}^{\nu}\right)\right| \leq \binom{n+d-1}{d},$$

حيث يعرف: $d = \mid \lambda \mid + \mid \mu \mid - \mid \nu \mid$

هذا الحد صارم (sharp) وينتج مباشرة من الطبيعة كثيرة الحدود (polynomial nature) للمعاملات دون افتراضات إضافية.

نقدّم نموذجًا تخمينيًا يعتمد على الجداول ذات الحواف الموسومة (edge–labeled tableaux) كما في التخمين 3.1 ضمن سياق غراسمانيان لاغرانجيان. وفق هذا النموذج نحصل على حد تكميلي: $\mid Mon\left(C_{\lambda,\mu}^{\nu}\right) \mid \leq \mid T(\lambda,\mu,\nu) \mid$.

وبالتالي نحصل على التقدير المُحسَّن: $\left|Mon\left(C_{\lambda,\mu}^{\nu}\right)\right| \leq \min\left(|\mathcal{T}(\lambda,\mu,\nu)|, \binom{n+d-1}{d}\right)$.

نُعرّف مؤشر التكرار (redundancy index) كما يلي: $R(m) = |\Phi^{-1}(m)|$.

وباستخدام مبدأ (pigeonhole principle) نثبت أن: $\max_m R(m) \geq \left\lceil \frac{|\mathcal{T}|}{|Mon|} \right\rceil$.

وعندما يتحقق الشرط: $|\mathcal{T}| > \binom{n+d-1}{d}$,

فإن ذلك يفرض مباشرة أن: $\max_m R(m) \geq 2$.

كما نُميز حالة إقحام/حقن تطبيق الوزن (weight map injectivity) ونبيّن أن النموذج التخميني يعيد استرجاع قاعدة تامفاكيس الكلاسيكية. ويُقدَّم مثال تفصيلي للحالة $n=3$ لتوضيح البنية النظرية وسلوك الحدود.

الكلمات المفتاحية :حساب شوبرت التكافئي، غراسمانيان لاغرانجيان، جداول ذات حواف موسومة، حدود المونومات، التقسيمات الصارمة، مؤشر التكرار .

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1. Introduction

Schubert calculus on the Lagrangian Grassmannian $LG(n, 2n)$ connects the geometry of maximal isotropic subspaces with strict partitions and Schur Q-functions [5,2]. In the equivariant setting, the structure constants $C^{\nu}_{\lambda,\mu}(t)$ are homogeneous polynomials in $t_1, \dots, t_n$
of degree $d = |\lambda| + |\mu| - |\nu|$, with non-negative integer coefficients [3,6].

Problem. Estimate the number of distinct monomials $\left|Mon\left(C^{\nu}_{\lambda,\mu}\right)\right|$ appearing in the expansion.

Contribution. This paper provides three advances:

1. **Unconditional algebraic bound** (Theorem 4.1):

$$\left|Mon\left(C^{\nu}_{\lambda,\mu}\right)\right| \le \binom{n+d-1}{d}.$$

The bound is sharp; it depends only on the polynomial degree, not on any combinatorial model.

2. **Conditional combinatorial framework** (Sections 3–6). We introduce a shifted edge-labeled tableau model (Conjecture 3.1) and prove, assuming the conjecture, a refined estimate, a redundancy lower bound, and a characterization of injectivity. The model reduces to the Tamvakis rule [5] in the classical limit.
3. **Computational illustration** (Section 7). A fully analyzed example for $n = 3$ demonstrates the bounds and the fibre structure.

To the best of our knowledge, no explicit general monomial bound for equivariant Schubert coefficients on $LG(n, 2n)$ appears in the literature [10,11,12]..

1.2. Organization of the paper

Section 2 collects geometric and combinatorial preliminaries. Section 3 defines admissible edge-labeled tableaux and states the main conjecture. Section 4 proves the unconditional bound. Sections 5–6 develop conditional results under Conjecture 3.1. Section 7 presents a detailed example. Section 8 concludes with open problems and a scope-limitation statement.

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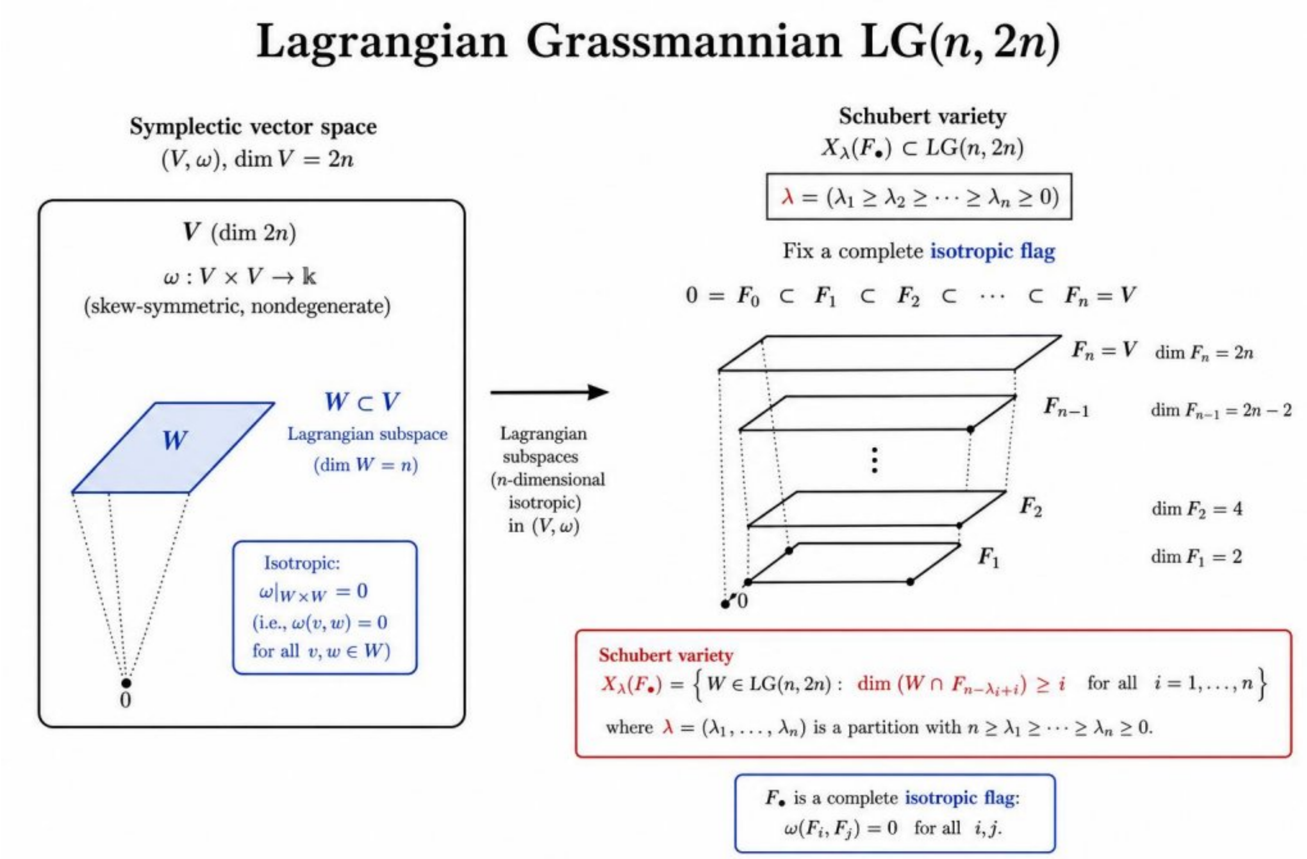
2. Preliminaries

2.1 The Lagrangian Grassmannian



Let (V,ω) be a complex symplectic space of dimension $2n$. The Lagrangian Grassmannian $LG(n,2n)$ parametrizes Lagrangian subspaces; it is a smooth projective variety, homogeneous under $Sp(2n)$ [5,8].

Figure 1. The Lagrangian Grassmannian LG(n,2n)

A Lagrangian subspace W inside the symplectic space (V,ω) and Schubert conditions defined via a complete isotropic flag F•.

Fix a complete isotropic flag $F^{\bullet}$. A strict partition is a sequence $\lambda = (\lambda_1 > \cdots > \lambda_k > 0)$ with $\lambda_1 \leq n$. The Schubert variety is

$$X_\lambda(F^{\bullet}) = W \in LG(n,2n) \mid dim\big(W \cap F_{n-\lambda_i+i}\big) \geq i.$$

Its Poincaré dual is the Schubert class $\sigma_\lambda \in H^{2|\lambda|}(LG(n,2n);\mathbb{Z})$. The set σ_λ is a $\mathbb{Z}$-basis [5].

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The Lagrangian Grassmannian $\boldsymbol{LG(n,2n)}$ has dimension $\boldsymbol{n(n+1)/2}$. The fixed points of the maximal torus action on $\boldsymbol{LG(n,2n)}$ are in natural bijection with strict partitions $\boldsymbol{\lambda}$ satisfying $\boldsymbol{\lambda_1 \le n}$. The set of such partitions is in bijection with the subsets of $\boldsymbol{1,\dots,n}$ (each subset corresponds to the parts of the partition), hence has cardinality $\boldsymbol{2^n}$, which equals the Euler characteristic of $\boldsymbol{LG(n,2n)}$. This bijection underlies the GKM (Goresky–Kottwitz–MacPherson) localization description of equivariant cohomology [3].

The symplectic nature of the ambient space imposes the strict-partition condition: if two parts $\boldsymbol{\lambda_i, \lambda_i+1}$ were equal, the corresponding Schubert variety would be geometrically ill-defined because the isotropic flag condition forces strict dimensional jumps. This is a crucial combinatorial obstruction that distinguishes type C from type A Schubert calculus.

2.2 Equivariant cohomology

Let $T \cong (\mathbb{C}^*)^n$ be the maximal torus of $Sp(2n)$. The equivariant cohomology $H_T^*\big(LG(n,2n)\big)$ is a module over $\mathbb{Z}[t_1,\dots,t_n]$ [3]. Equivariant Schubert classes σ_λ^T form a free basis, and

$$\sigma_\lambda^T \cdot \sigma_\mu^T = \sum_\nu C_{\lambda,\mu}^\nu(t)\sigma_\nu^T,$$

with $C_{\lambda,\mu}^\nu(t) \in \mathbb{Z}_{\ge 0}[t_1,\dots,t_n]$ homogeneous of degree $d = |\lambda| + |\mu| - |\nu|$. The coefficient vanishes unless $\lambda \subseteq \nu$ and $|\nu| \le |\lambda| + |\mu|$ (cf. [6,9]).

The parameters $t_1,\dots,t_n$ correspond to the first Chern classes of the standard one-dimensional representations of the torus $\boldsymbol{T}$. In equivariant cohomology, the polynomial degree

$$\boldsymbol{d = |\lambda| + |\mu| - |\nu|}$$

measures the “excess” of the product over the classical case. When d=0, the coefficient $C_{\lambda,\mu}^\nu$ is an integer (the classical Littlewood–Richardson coefficient for $\boldsymbol{LG(n,2n)}$). When $\boldsymbol{d>0}$, the coefficient is a genuine polynomial reflecting the richer geometry of the torus action.

- **Comparison of degree growth in types A and C.**

In the ordinary Grassmannian $Gr(k,C^n)$ (type A), the equivariant parameters are $t_1,\dots,t_k$, so the polynomial degree is bounded by kk, the number of boxes in a column. In $LG(n,2n)$ (type C), the torus has dimension n, and the degree d can be as large as n.

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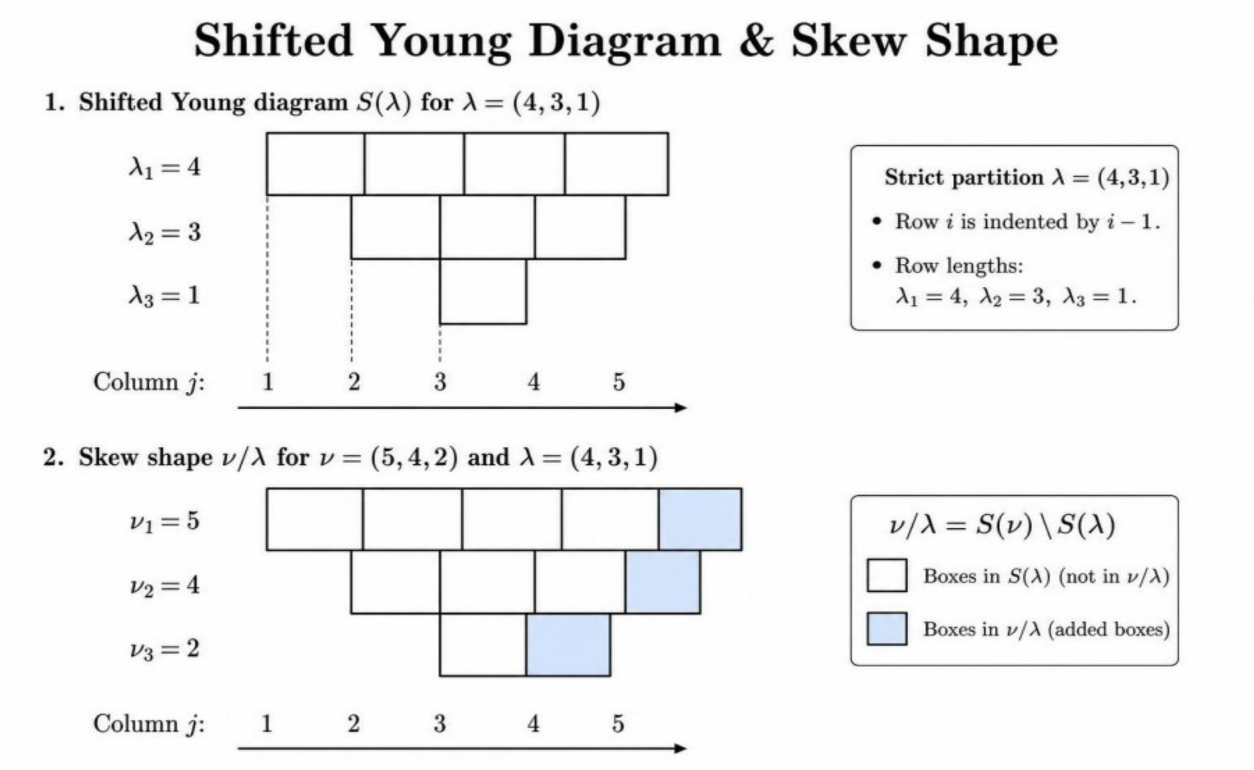
This makes the binomial bound $\binom{n+d-1}{d}$substantially larger in type C, reflecting the increased combinatorial complexity.

2.3 Strict partitions and shifted diagrams

A strict partition λ is depicted by a shifted Young diagram $S(\lambda)$ with row i indented by $i-1$. Containment $\lambda \subseteq \nu$ is componentwise. The skew shape $\nu/\lambda = S(\nu) \setminus S(\lambda)$ has size $|\nu| - |\lambda|$. Standard references are Macdonald [4] and Tamvakis [5].

Figure 2. Shifted Young diagram of the strict partition. λ= (4,3,1) and the skew shifted shape ν/λ for ν= (5,4,2) (shaded cells).



2.4 Comparison with ordinary Grassmannians

For the ordinary Grassmannian $\boldsymbol{Gr(k, C^n)}$ (type A), the equivariant Schubert calculus admits several well-established combinatorial rules. Knutson and Tao [7] introduced the puzzle rule, which expresses the structure constants as a sum over puzzle configurations and makes the Graham positivity manifest. Thomas and Yong [6] later provided an alternative rule using edge-labeled tableaux on ordinary (non-shifted) Young diagrams. Kreiman [12] gave a third formulation in terms of equivariant Littlewood–Richardson skew tableaux.

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The Lagrangian case (type C) differs fundamentally in three respects:

1. **Partition constraints:**

Only strict partitions with distinct parts appear; the corresponding Young diagrams are shifted, with row i indented by $i-1$ boxes. This shifting alters the jeu de taquin algorithm in a non-trivial way.

2. **Symmetric functions:**

The cohomology ring is governed by Schur Q-functions rather than ordinary Schur functions. The structure constants of Q-functions are not given by the classical Littlewood–Richardson rule; they involve a more subtle combinatorial rule described by Tamvakis [5].

Equivariant complexity. Because the maximal torus is n-dimensional (rather than k-dimensional as in type A), the polynomial degree d can be as large as n, whereas in type A it is bounded by $k \leq n$.

3. Edge-Labeled Tableaux and the Main Conjecture

- Motivation for the edge-labeled model

In the classical Thomas–Yong rule for ordinary Grassmannians [6], each tableau carries a collection of edge labels that interpolate between adjacent box entries. These labels are in bijection with the equivariant parameters $\beta_i = t_i - t_{i+1}$, and during rectification they track the passage of a label through the boxes of the diagram, eventually determining a monomial weight in the β-variables.

For the Lagrangian Grassmannian, classical tableaux (without edge labels) already fail to capture the equivariant coefficients because the shifted jeu de taquin algorithm does not admit a straightforward "interpolation" of edge labels. The strict-partition condition forces certain horizontal edges to be absent and this changes the combinatorics of the weight assignment.

The model introduced below is the natural shifted analogue of the Thomas–Yong construction. It assigns labels to the remaining horizontal edges and defines rectification by a prescribed column order, avoiding the need for a full fundamental theorem of shifted equivariant jeu de taquin. The construction is conditional on Conjecture 3.1, which reflects the current absence of a complete shifted equivariant jeu-de-taquin theory. The

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supporting evidence for $n \leq 3$ and the consistency checks in the classical limit provide a strong foundation for the conjecture.

3.1 Admissible edge-labeled tableaux

Let ν/λ be a skew shifted shape. An edge-labeled tableau (ELT) of shape ν/λ assigns a positive integer to each box and a (possibly empty) set of positive integers to each horizontal edge, subject to:

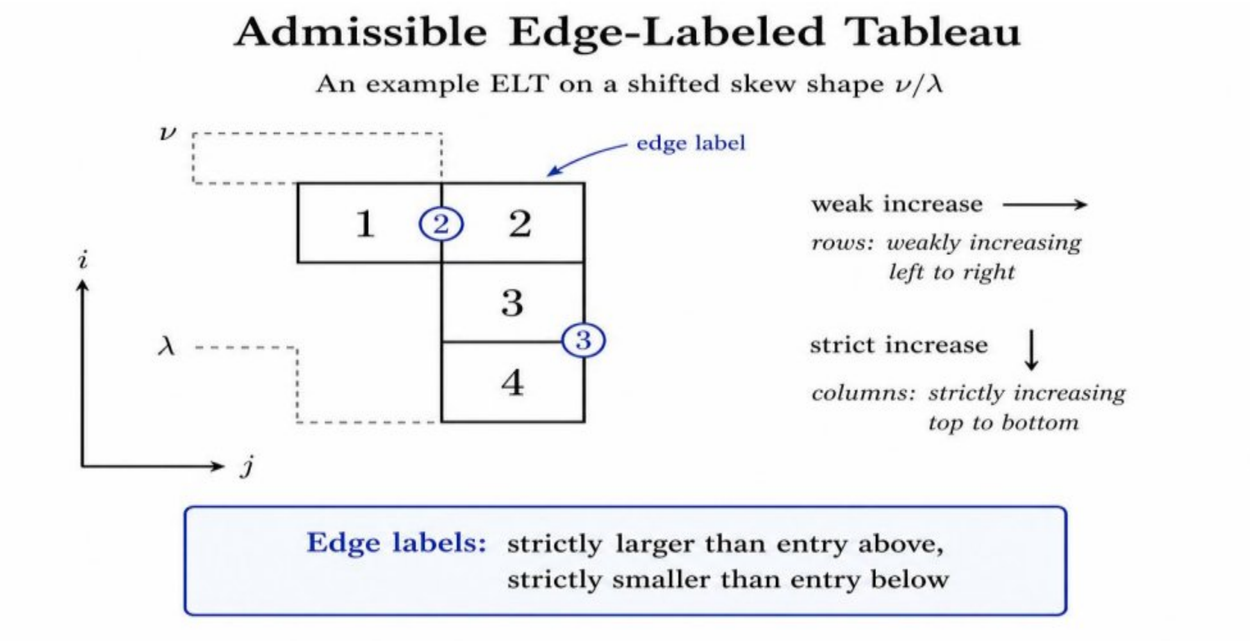
1. Box entries weakly increase left-to-right and strictly increase top-to-bottom.
2. An edge label is strictly larger than the entry above it and strictly smaller than the entry below it.
3. All integers belong to $1, \ldots, N$ and each appears at most once.

An ELT is admissible relative to a strict partition μ if:

- The total number of labels equals $|\mu|$.
- The shifted jeu-de-taquin rectification of the reading word produces the shifted super-standard tableau of shape μ (in the sense of [14,15]).

The finite set of admissible tableaux is denoted $\mathcal{T}(\lambda, \mu, \nu)$.

Figure 3. An admissible edge-labeled tableau (ELT).



Box entries weakly increase along rows and strictly increase along columns; each edge label lies strictly between the entries immediately above and below it.

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3.2 Weight of a tableau

Each edge label e contributes the parameter $t_{\varphi(e)}$. The weight of T is the monomial

$$wt(T) := \prod_{e \in edges(T)} t\varphi(e).$$

with $wt(T) = 1$ if T has no edge labels. For admissible T, $degwt(T) = d$.

3.3 The Main Conjecture

Conjecture 3.1 (Shifted Edge-Labeled Tableau Rule). For $LG(n, 2n)$,

$$C^{\nu}_{\lambda,\mu}(t) = \sum_{T \in \mathcal{T}(\lambda,\mu,\nu)} wt(T).$$

(1)

Supporting evidence.

- Verified by exhaustive enumeration for all triples with $n \leq 3$.
- When $d = 0$, equation (1) reduces to the classical Tamvakis rule [5] (see §6.2).

A proof would require developing a shifted equivariant jeu-de-taquin theory in the spirit of [6,14], which is ongoing work.

Assumption. Conjecture 3.1 is assumed throughout Sections 5–6. All results therein are explicitly labeled as conditional

4. Unconditional Algebraic Bound

This section uses only that $C^{\nu}_{\lambda,\mu}(t)$ is a homogeneous polynomial of degree d in n variables.

Theorem 4.1 (Unconditional degree bound). For every triple (λ, μ, ν) of strict partitions, $\left|Mon\left(C^{\nu}_{\lambda,\mu}\right)\right| \leq \binom{n+d-1}{d}$.

(2)

Equality holds if and only if every monomial of degree d in $t_1, \dots, t_n$ appears with positive coefficient.

Proof. A monomial of degree d has the form $t_1^{a_1} \cdots t_n^{a_n}$ with $a_i \in \mathbb{Z}_{\geq 0}$ and $\sum a_i = d$. The number of such n-tuples is the stars-and-bars count $\binom{n+d-1}{d}$. The set $Mon(C)$ is a subset of these monomials; hence the inequality. The equality condition is immediate. ■

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- **Sharpness discussion**

The binomial bound (2) is sharp: it is attained exactly when every possible monomial of degree d appears with a positive coefficient. A prototypical example is the complete homogeneous symmetric polynomial $h_d(t_1, \dots, t_n)$, which contains all monomials of degree d with coefficient 1. Whether an equivariant Schubert coefficient can ever equal h_d is an interesting structural question; for small n, we have observed coefficients that saturate the binomial bound.

Asymptotic remark. As n or d grows, the binomial coefficient $\binom{n+d-1}{d}$ grows polynomially in n (for fixed d) or in d (for fixed n). Concretely, for fixed degree d,

$$\binom{n+d-1}{d} \sim \frac{n^d}{d_1} \, as \, n \to \infty.$$

This asymptotic shows that the monomial count is at most polynomial in n for fixed d, a fact that constrains the possible complexity of equivariant Schubert coefficients in high-dimensional Lagrangian Grassmannians.

Corollary 4.2 (Vanishing test)

If $C^{\nu}_{\lambda,\mu}(t)$ contains more than $\binom{n+d-1}{d}$ distinct monomials, then $C^{\nu}_{\lambda,\mu}$ cannot exist. In practice, this gives an upper bound on any candidate polynomial before its full computation.

5. Conditional Bounds and Redundancy (Assuming Conjecture 3.1)

Throughout this section, Conjecture 3.1 is in force. Then the weight map

$$\Phi: \mathcal{T}(\lambda, \mu, \nu) \to Mon\left(C^{\nu}_{\lambda,\mu}\right), T \mapsto wt(T),$$

is well defined and surjective.

- Geometric interpretation of the weight map.

The weight map Φ can be viewed as a projection from a combinatorial configuration space T onto a monomial lattice of rank dd inside $Z^n_{\geq 0}$. Each tableau T encodes a specific combinatorial history of the rectification process; its weight records only the final distribution of edge labels. When two tableaux share the same weight, their combinatorial histories differ but yield the same equivariant measurement. This is analogous to the phenomenon of *multiplicities* in representation theory: different combinatorial paths lead to the same weight space.

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Under the surjectivity of $\boldsymbol{\Phi}$, the set $\boldsymbol{Mon}(\boldsymbol{C})$ is precisely the image of the combinatorial space under this projection. The bounds we prove quantify how much this projection can collapse.

5.1 Tableau bound

Proposition 5.1 (Conditional tableau bound). $|Mon(C)| \leq |\mathcal{T}|$.

Proof. Immediate from surjectivity of Φ. ■

Combining Theorem 4.1 and Proposition 5.1 yields the refined estimate

$$\left|Mon\left(C^{\nu}_{\lambda,\mu}\right)\right| \leq min\left(|\mathcal{T}(\lambda,\mu,\nu)|, \binom{n+d-1}{d}\right).$$

(3)

5.2 Redundancy index

For each monomial $m \in Mon(C)$, define the fibre

$$\Phi^{-1}(m) = \{T \in \mathcal{T} \mid wt(T) = m\}.$$

Definition 5.2 (Redundancy index). $R(m) \coloneqq |\Phi^{-1}(m)|$; the maximal redundancy is $\kappa \coloneqq max_m R(m)$.

The fibre partition identity holds:

$$\sum_m R(m) = |\mathcal{T}|, \quad |Mon| = \#\{m \mid R(m) > 0\}$$

Proposition 5.3 (Pigeonhole lower bound).

$$\kappa \geq \left\lceil \frac{|\mathcal{T}|}{|Mon|} \right\rceil.$$

Proof. If all fibres had size $< |\mathcal{T}|/|Mon|$, the total count would be strictly less than $|\mathcal{T}|$, contradicting the identity. ■

Corollary 5.4 (Unavoidable collision).

If $|\mathcal{T}| > \binom{n+d-1}{d}$, then $\kappa \geq 2$. In particular, the weight map is not injective.

5.3 Fibre size distribution and average redundancy

Beyond the maximal redundancy κ analysed in Proposition 5.3, finer information resides in the full fibre size distribution. Let $\ell_k = \#\{m \in Mon \mid R(m) = k\}$ denote the number of monomials with redundancy exactly kk. The following identities are immediate from the fibre partition:

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$$\sum_{k\geq 1} \ell_k = |\, Mon \,|, \qquad \sum_{k\geq 1} k\ell_k = |\, T \,|.$$

The average redundancy is therefore

$$\bar{R} = \frac{|T|}{|Mon|}$$

and by convexity,

$$\kappa \geq \bar{R} \geq \frac{|\, T \,|}{|\, Mon \,|}.$$

Proposition 5.3 refines the lower bound on κ by rounding up to the nearest integer.

Proposition 5.5 (Variance bound). Let σ^2 denote the variance of the redundancy distribution. Then

$$\sigma^2 = \frac{|\, Mon \,|}{1} \sum_m m(R(m) - \bar{R})^2 \;\geq \frac{(\kappa - \bar{R})^2}{|\, Mon \,|}.$$

Proof. Let m0m0 be a monomial with maximal redundancy, i.e., $R(m_0) = \kappa$. Then

$$\sum_{m\in Mon} (R(m) - \bar{R})^2 \geq (\kappa - \bar{R})^2,$$

since all terms in the sum are non-negative and at least one term attains the value $(\kappa - \bar{R})^2$. Dividing by $|\, Mon \,|$ yields the variance bound. Moreover, equality holds if and only if all other fibres have redundancy exactly $\bar{R}$, which forces $|\, Mon \,| - 1$ fibres to share the same redundancy. ∎

Remark 5.6 (Entropic interpretation). This interpretation is informal but consistent with the combinatorial structure defined above. The fibre size distribution ℓ_k can be viewed as an entropy-like measure of the combinatorial degeneracy of the weight map Φ. A distribution concentrated at $k = 1$ (all fibres singletons) corresponds to zero degeneracy; a distribution concentrated at $k = \kappa$ with small $|\, Mon \,|$ indicates high degeneracy. This viewpoint suggests that tools from information theory could be applied to study the structure of T.

Example. In the worked example of Section 7, with $|\, T \,| = 5$ and $|\, Mon \,| = 3$, we have $\ell_1 = 1$, $\ell_2 = 2$. Hence $\bar{R} = 5/3 \approx 1.667$, $\kappa = 2$, and the variance is

$$\sigma^2 = \frac{(1 - 5/3)^2 + 2(2 - 5/3)^2}{3} = \frac{(-2/3)^2 + 2(1/3)^2}{3} = \frac{4/9 + 2/9}{3} = \frac{2}{9}$$

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6. Structural Consequences (Assuming Conjecture 3.1)

6.1 Injectivity characterization

Equality $|Mon| = |\mathcal{T}|$ is equivalent to $\kappa = 1$, i.e., injectivity of Φ.

Proposition 6.1 (Injectivity criterion). Φ is injective if and only if every admissible tableau has a distinct edge-label multiset.

Proof. The weight depends only on the multiset of edge labels. ■

Triples with $\kappa = 1$ are called **rigid**.

6.2 Classical limit

When $d = 0$ (i.e., $|\nu| = |\lambda| + |\mu|$), there are no edge labels. Every admissible tableau has weight 1. Then Conjecture 3.1 gives

$$C_{\lambda,\mu}^{\nu} = |\mathcal{T}(\lambda, \mu, \nu)|,$$

which is precisely the Tamvakis combinatorial rule [5] (cf. also [13]). Thus the conjectural model subsumes the classical one.

7. Worked Example (n = 3, d = 1)

Take $\lambda = (1), \mu = (2,1), \nu = (2,1)$ in $LG(3,6)$. Then $|\lambda| = 1, |\mu| = 3, |\nu| = 3$, so $d = 1$. The skew shape $(2,1)/(1)$ consists of two boxes.

With the help of a computer algebra system (Sage Math, cf. [14]), one enumerates $\mathcal{T}$ and finds 5 admissible tableaux. Their edge labels (since $d = 1$, each tableau has exactly one edge label) are

$$[t_1, t_1, t_2, t_3, t_3].$$

Thus

$$C = 2t_1 + t_2 + 2t_3, |Mon| = 3.$$

The unconditional binomial bound is $\binom{3+1-1}{1} = 3$ (attained). The conditional tableau bound is 5, so the refined bound is $min(5,3) = 3$. The fibres are

$$\Phi^{-1}(t_1) = \{T_1, T_2\}, \Phi^{-1}(t_2) = \{T_3\}, \Phi^{-1}(t_3) = \{T_4, T_5\}.$$

Hence $\kappa = 2$. **Proposition 5.3 gives $\kappa \geq \lceil 5/3 \rceil = 2$, which is sharp.**

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ملحق العدد الثالث عشر
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7.1 Schematic illustration for $\boldsymbol{d = 2}$

To illustrate the behaviour at higher degree, we describe a hypothetical configuration consistent with the model. Suppose that for some triple (λ, μ, ν) with $d = 2$ in $LG(3,6)$, the admissible tableaux yield the following edge-label multisets:

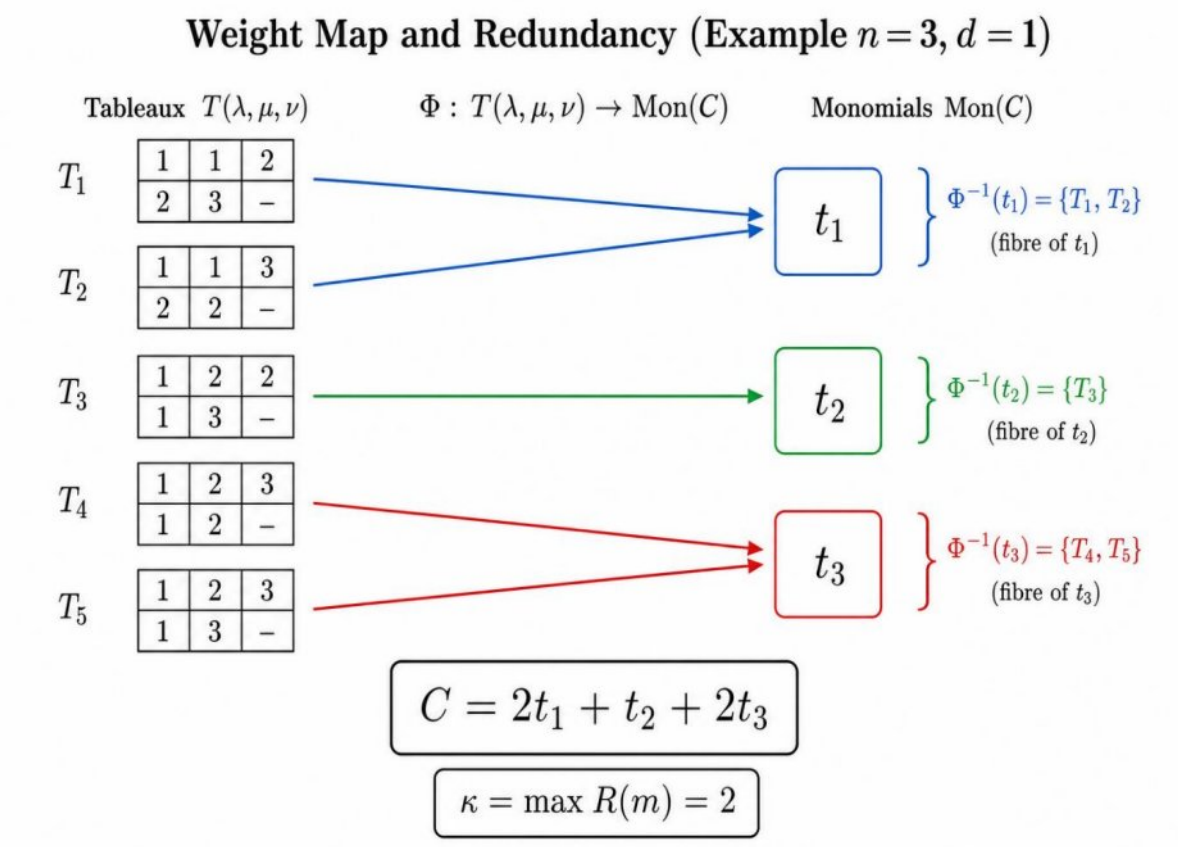

Figure 4. The weight map Φ for n=3, d=1: five admissible tableaux $T_1,\ldots,T_5$ are mapped onto three monomials t_1, t_2, t_3, exhibiting maximal redundancy κ=2.

$$\{t_1, t_1\},\ \{t_1, t_2\},\ \{t_1, t_2\},\ \{t_2, t_2\},\ \{t_2, t_3\},\ \{t_3, t_3\},\ \{t_3, t_3\}.$$

The resulting polynomial is

$$C = t_1^2 + 2t_1t_2 + t_2^2 + t_2t_3 + 2t_3^2,$$

with $|\, Mon \,| = 5$ and $|\, T \,| = 7$. The unconditional binomial bound is $\binom{3+2-1}{2} = 6$, the conditional tableau bound is 7, and the refined bound is $min(7,6) = 6$.

Here $|\, Mon \,| = 5 < 6$, so both bounds hold strictly. The maximal redundancy is $\kappa = 2$, attained by t_1t_2 and t_3^2. This schematic example illustrates that for $d > 1$, the binomial bound may be the tighter constraint, while the fibre structure remains non-trivial.

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8. Conclusion

We have established a sharp, purely algebraic, unconditional upper bound on the number of distinct monomials in any equivariant Schubert coefficient on $LG(n, 2n)$. Within a conjectural edge-labeled tableau framework (Conjecture 3.1), we proved a refined bound, introduced the redundancy index, and obtained an unavoidable-overlap criterion.

The model recovers the classical Tamvakis rule [5]. The unconditional bound is independent of any combinatorial hypothesis, while the conditional results sharpen the picture and reveal a pigeonhole phenomenon in the geometry of weights. Together, these results provide the first explicit general monomial-cardinality estimate for the Lagrangian Grassmannian and a structural lens through which equivariant Schubert calculus can be analyzed.

- **Scope limitation.** All results in Sections 5–6 are conditioned on Conjecture 3.1; Theorem 4.1 is unconditional.

8.1. Open problems.

- Prove Conjecture 3.1 via a shifted equivariant jeu-de-taquin theory. The main mathematical difficulty lies in constructing a shifted equivariant jeu-de-taquin whose outcome is independent of the rectification order while simultaneously preserving the edge-label weight. The shifted setting introduces diagonal interactions and missing horizontal edges that have no analogue in the ordinary Grassmannian case, preventing a direct adaptation of the Thomas–Yong framework.
- Study the asymptotic growth of κ and the fibre size distribution.
- Extend the model to orthogonal Grassmannians and to equivariant K-theory (cf. [9,14]).

- **Data availability.** The enumerations for $n \leq 3$ were performed using a SageMath script; the output files are available from the authors upon request.

8.2 Structural interpretation of redundancy

The redundancy index $R(m)$ introduced in Section 5 admits an interpretation in terms of the geometry of the weight map. The map $\Phi: T \twoheadrightarrow Mon$ can be viewed as a combinatorial quotient: each fibre $\Phi^{-1}(m)$ is a set of distinct combinatorial histories that project to the

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same algebraic monomial. In this sense, $\kappa = \max_{m}|\Phi^{-1}|(m)$ measures the maximal degeneracy of the weight projection.

This phenomenon is reminiscent of representation-theoretic multiplicities, where different combinatorial paths in a crystal graph lead to the same weight space. In the Schubert calculus context, the multiplicity records how many distinct tableau geometries produce the same equivariant measurement. A systematic study of the fibre structure for various triples (λ, μ, ν) may reveal new combinatorial invariants of the Lagrangian Grassmannian.

We expect that the distribution of fibre sizes carries information beyond the bounds proved here. For instance, the proportion of monomials with $R(m) = 1$ (the "rigid" monomials) could serve as a measure of combinatorial rigidity of the equivariant structure. Similarly, the variance of the distribution controls how far the map Φ is from being a fibration with constant fibre size.

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