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Applications of Residue Theory in the Partial Fraction Expansion of Complex Rational Functions

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Abstract

In this paper, we use residue theory to perform a partial fraction expansion of a rational function represented as the quotient of two complex polynomials. While most literature focuses on the decomposition of real polynomials, extending this to complex polynomials is significantly more detailed and intricate.

We begin by explaining the basic concepts of residue theory and its application in complex analysis. Subsequently, we demonstrate how this theory can be applied to identify the poles of a complex rational function, calculate the residues at these poles, and represent the expanding of a function in terms of these residues.

Several examples illustrate the method and provide evidence of its effectiveness. Our results clearly indicate that residue theory is an effective method for analyzing polynomial fractions in the complex plane. This work extends the understanding of partial fraction decompositions in complex domains and suggests broad potential applications in mathematical analysis and engineering.

Keywords: Residue Theory, Simple pole, Partial Fraction Decomposition Function of Complex variables.

الملخص:

في هذا العمل، نقدم طريقة فعالة لتحليل الكسور الجزئية للدوال الكسرية المعقدة، وبصورة خاصة التي يتم تمثيلها كنسبة بين كثيرات الحدود المركبة. تعتمد الطريقة على نظرية البقايا لتحديد وتفكيك أقطاب الدالة الكسرية في المستوى المركب.

تعمل هذه الطريقة على توسيع المفهوم التقليدي لتحليل الكسور الجزئية من المجال الحقيقي إلى المجال المركب، مما يوفر إطارًا جديدًا لتوسيع الدوال الكسرية المركبة. من خلال استخدام نظرية البقايا، نقدم تقنية مختلفة لتمثيل تحليل الدوال الكسرية المعقدة، والتي يمكن تطبيقها على مجموعة متنوعة من المسائل الرياضية والفيزيائية .

Introduction

The partial fraction decomposition of rational functions is a basic technique of mathematical analysis [4]; it is relevant to solving differential equations, executing integrals, and simplifying expressions in many fields, such as mathematics and engineering. However, although most traditional partial-fraction decomposition methods regard real polynomials exclusively, challenges and opportunities lie in their extension to complex polynomials.

Residue theory is a powerful tool for complex analysis and provides an elegant way to analyze complex rational functions [10]. Developed at the beginning of the 20th century, the theory of residues describes the behavior of complex functions near their singular points and provides a general rule for evaluating integrals that involve complex variables[1]. With the help of residues, it is possible to split cumbersome polynomial fractions into simple partial fractions that are easy to handle.

This study investigates the use of residue theory in the partial-fraction decomposition of the quotients of two complex polynomials [8]. We provide an overview of residue theory by describing its relationship with partial fraction decomposition.

Then, we describe how residue theory can be applied to solve the given problem, including pole location, computation of residues [6][7], and reconstruction of the partial fraction expansion. We illustrate by examples how complicated polynomial fractions can be very considerable simplified using residue theory [10]. Results indicate the usefulness of this approach for a deeper understanding of complex rational functions and also provide an application in theoretical and practical contexts [3]. In other words, this paper generalizes the traditional methods of partial fractions into complex polynomials and enhances the practice of residue theory within mathematical analysis.

2. Method

2.1 Functions of complex variables

The function of the complex variables z is a rule that assigns to each value z in a set D one and only one complex value w.

$w = f(z)$

Just as the complex variable z can be delineated by its constituent real and imaginary components, expressed as, $z = x + iy$, we denote $w = u + iv$, where u and v

represent the real and imaginary components of w, respectively. This formulation then gives us

$f(x+iy)=u+iv$. Since u and v are functions of the variables x and y, they may be regarded as real valued functions of the real variables x and y; in particular, $u=u(x,y)$and

$v=v(x,y)$. In view of the above, the composition function can be written as

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$

where u and v are two prescribed real-valued functions of the real variables$x$ and y. [11]

Analyticity of Functions 22.

2.2.1 Analyticity at a Point

A complex function $w=f(z)$ is said to be analytic at a point z_0 if it is differentiable at z_0 and at every point in a neighborhood around z_0 .

In other words, $f(z)$ must satisfy the condition of differentiability in an open set that contains z_0.

2.2.2 Analyticity in a Domain

A function f is considered analytic in the domain D if it is analytic at every point within D. When a function is analytic throughout the entire domain D, it is often referred to as holomorphic or regular. This property implies that the function is infinitely differentiable and can be locally expressed as a convergent power series.[**8**]

2.3 Laurent Expansion Theorem

The Laurent series is an expansion of a complex function $f(z)$ about a singularity. If $f(z)$ is analytic in an annular region around a point z_0, that is, within $r_1 <| z-z_0 |< r_2$, then $f(z)$ can be expressed as a Laurent series in the following form :

$f(z)=\sum_{-\infty}^{\infty}(z-z_0)^n$, where a_n are the coefficients of the series, and they can be computed

by :

$$a_n=\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Here, γ is a closed contour around the point z_0 within the annular region. [3]

2.3.1 Singularities of Complex Functions

In the theory of complex functions, singularities are points where a function fails to be analytic. Singularities can be classified based on the behavior of the Laurent series expansion around the singular point. The following table gives a summary of the types of singularities and their corresponding Laurent series expansions around a point $z = z_0$.[8]

$z = z_0$	Laurent Series for $0 < \|z - z0\| <$
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots\dots\dots\dots$
Pole of order n	$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \dots\dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots\dots$
Simple pole	$\frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots\dots\dots\dots$
Essential singularity	$\dots\dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots\dots\dots\dots$

Table 1 . Forms of Laurent series.

Theorem 1 [11]

A function $f(z)$ analytic in the punctured disk $D_R^*(z_0) = \{z: 0 < |z - z_0| < R\}$ has a pole of order k at $z = z_0$ if and only if f can be expressed in the of $f(z) = \frac{h(z)}{(z-z_0)^k}$ where $h(z)$ is analytic at $z = z_0$ and $h(z) \neq 0$.

Residue Explanation 2.4

The residue of a function $f(z)$ at a singularity z_0 is the coefficient a_{-1}in the Laurent series expansion of $f(z)$ around z_0.This can directly be taken out of the Laurent series and is given as:

$$\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i}\oint_{\gamma}\frac{f(x)}{(z-z_0)}dz$$

The Residue Theorem states that if the function $f(z)$ is analytic inside and on a simple closed contour C, except for a finite number of singularities $z_1, z_2, \dots, z_n$ inside C, then the following is the value of the integral of $f(z)$ around :

$$\oint_C f(z)dz = 2\pi i\sum_{k=1}^{n}\operatorname{Res}(f, z_k)$$

The Laurent expansion is a mandatory tool in the case where such functions have singularities. In the case of this function represented in the form of a Laurent series, the residue, $Res(f, z_0)$, will be the coefficient in front of $(z - z_0)^{-1}$.Hence, the Laurent expansion becomes quite fundamental within the frame of evaluating a number of complex integrals with the help of the residue theorem.[5]

2.5 Partial Fraction Expansion

Partial fraction expansion decomposes a rational function into a sum of simpler fractions, typically for purposes of integration, solving differential equations, or simplifying inverse Laplace or Fourier transformations. Residue theory provides an elegant and systematic method to find the coefficients in these expansions, particularly for functions involving complex poles.

Consider a rational function :

$$F(z) = \frac{P(z)}{Q(z)}$$

Where $P(z)$ and $Q(z)$ are polynomials, and $\deg(P(z)) < \deg(Q(z))$. The function $F(z)$ can be decomposed into a sum of partial fractions, where each fraction corresponds to a root (pool) of the denominator $Q(z)$.

The general form of the partial fraction expansion is :

$$F(z) = \sum_{k=1}^{n} \frac{A_k}{z - z_k}$$

Where z_k are the distinct poles(root of $Q(z) = 0$) and A_k are constants that need to be determined .

3. Result

3.1 Using Residue Theory to Find the Coefficients

The coefficients A_k can be found directly by calculating the residues of the functions $F(z)$ at their poles using residue theory.

3.1.1 Simple poles

The residue of a function $F(z)$ at a simple pole $z = z_k$ is given by :

$$\lim_{z \to z_k} (z - z_k)F(z) \; . \; A_k = \operatorname{Re} s(F(z), z_k) =$$

Thus, for each pole z_k of $Q(z)$, the coefficient A_k in the partial fraction expansion is simply the residue of $F(z)$ at z_k.

Lemma 3.1.1

Let $P(z)$ be a polynomial of degree at most 2 .If a, b and c are distinct complex numbers, then [9]

$$F(z) = \frac{P(z)}{(z-a)(z-b)(z-c)} = \frac{A_1}{z-a} + \frac{A_2}{z-b} + \frac{A_3}{z-c}$$

Where

$$A_1 = \operatorname{Res}[F, a] = \frac{P(a)}{(a-b)(a-c)}$$

$$A_2 = \operatorname{Res}[F, b] = \frac{P(b)}{(b-a)(b-c)}$$

$$A_3 = \operatorname{Res}[F, c] = \frac{P(c)}{(c-a)(c-b)}$$

Example 1

One need calculate the coefficients of partial fractions of the function

$$F(z) = \frac{3z+2}{z(z-1)(z-2)}$$

Solution

$$F(z) = \frac{3z+2}{z(z-1)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$$

Using the residue theory .

$$A_1 = Res[F, 0] = \lim_{z \to 0} zF(z) = \lim_{z \to 0} \frac{3z+2}{(z-1)(z-2)} = 1$$

$$A_2 = Res[F, 1] = \lim_{z \to 1} (z-1)F(z) = \lim_{z \to 1} \frac{3z+2}{z(z-2)} = -5$$

$$A_3 = Res[F, 2] = \lim_{z \to 2} (z-2)F(z) = \lim_{z \to 2} \frac{3z+2}{z(z-1)} = 4$$

Therefore:

$$F(z) = \frac{3z+2}{z(z-1)(z-2)} = \frac{1}{z} - \frac{5}{z-1} + \frac{4}{z-2}$$

3.1.2 Higher-Order Poles

If the rational function has higher-order poles (1.e,the pole at z_k has multiplicity m), the partial fraction expansion includes terms like :

$$\frac{A_1}{(z-z_k)}+\frac{A_2}{(z-z_k)^2}+\dots\dots+\frac{A_m}{(z-z_k)^m}$$

In this case, residue theory can still be used to find the coefficients $A_1, A_2,\dots\dots\dots, A_m$. The coefficient A_j corresponding to the $j-th$ term at a pole of order m can be computed as:

$$A_j=\frac{1}{(m-j)!}\lim_{z\to z_k}\frac{d^{m-j}}{dz^{m-j}}[(z-z_k)^m F(z)]$$

Example 2

One need calculate the coefficients of partial fractions of the function

$$F(z)=\frac{z^2-7z+4}{z^2(z+4)}$$

Solution

$$F(z)=\frac{z^2-7z+4}{z^2(z+4)}=\frac{A_1}{z}+\frac{A_2}{z^2}+\frac{B}{z+4}$$

Using the residue theorem:

$$\lim_{z\to 0}\frac{d}{dz}\frac{z^2-7z+4}{z+4}=\lim_{z\to 0}\frac{(z+4)(2z-7)-(z^2-7z+4)(1)}{(z+4)^2}=-2\quad A_1=\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}z^2F(z)=$$

$$A_2=\frac{1}{0!}\lim_{z\to 0}z^2F(z)=\lim_{z\to 0}\frac{z^2-7z+4}{z+4}=1$$

$$\mathrm{B}=\lim_{z\to -4}(z+4)F(z)=\lim_{z\to\to -4}\frac{z^2-7z+4}{z^2}=3$$

Therefore :

$$F(z)=\frac{-2}{z}+\frac{1}{z^2}+\frac{3}{z+4}$$

Example 3

One need calculate the coefficients of partial fractions of the function

$$F(z)=\frac{2z^2-3z-1}{(z-1)^3}$$

Solution

$$F(z)=\frac{2z^2-3z-1}{(z-1)^3}=\frac{A_1}{z-1}+\frac{A_2}{(z-1)^2}+\frac{A_3}{(z-1)^3}$$

Using the residue theorem:

$$A_1=\frac{1}{2!}\lim_{z\to 1}\frac{d^2z}{dz^2}(z-1)^3F(z)=\frac{1}{2}\lim_{z\to 1}\frac{d^2z}{dz^2}(2z^2-3z-1)=2$$

$$A_2=\frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^3F(z)=\lim_{z\to 1}\frac{d}{dz}(2z^2-3z-1)=\lim_{z\to 1}(4z-3)=1$$

$$A_3=\lim_{z\to 1}(z-1)^3F(z)=\lim_{z\to 1}(2z^2-3z-1)=-2$$

Therefore:

$$F(z)=\frac{2z^2-3z-1}{(z-1)^3}=\frac{2}{z-1}+\frac{1}{(z-1)^2}-\frac{2}{(z-1)^3}$$

Conclusion

The theory of residues, applied to partial fraction expansion, represents one of the most powerful, but at the same time elegant techniques in the process of decomposing complex rational functions into fractions. This is particularly true in systems that have complex poles. In this regard, after computing the residues at the poles, one will be able systematically effectively and perform partial fraction expansions, which represent a very useful means in a multitude of mathematical and physical applications. Furthermore, residue theory allows us to evaluate complex integrals of the form $\oint_C f(z)dz$, just by summing the residues at the isolated singularities of the function f within a closed contour C. Consequently, it serves as a further illustration of the flexibility and power of residue theory in complex analysis.

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